Diffusion of test particles in stochastic magnetic fields for small Kubo numbers

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(Received 21 July 2005; published 8 February 2006)

Motion of charged particles in a collisional plasma with stochastic magnetic field lines is investigated on the basis of the so-called *A*-Langevin equation. Compared to the previously used *V*-Langevin model, here finite Larmor radius effects are taken into account. The *A*-Langevin equation is solved under the assumption that the Lagrangian correlation function for the magnetic field fluctuations is related to the Eulerian correlation function (in Gaussian form) via the Corrsin approximation. The latter is justified for small Kubo numbers. The velocity correlation function, being averaged with respect to the stochastic variables including collisions, leads to an implicit differential equation for the mean square displacement. From the latter, different transport regimes, including the well-known Rechester-Rosenbluth diffusion coefficient, are derived. Finite Larmor radius contributions show a decrease of the diffusion coefficient compared to the guiding center limit. The case of small (or vanishing) mean fields is also discussed.

DOI: 10.1103/PhysRevE.73.026404 PACS number(s): 52.25.Fi, 05.40.−a, 52.25.Gj, 52.35.Ra

I. INTRODUCTION

Plasma confinement due to magnetic fields is the basic concept for magnetic fusion. Mainly two lines, the tokamak and the stellarator, are being proposed at the moment to reach the goal of a well-confined burning plasma. Until the beginning of magnetic fusion research, the problem of particle and heat transport is in the focus of theoretical and experimental investigations. The reason for such an extraordinary interest in this problem lies in the unexpected large losses caused by anomalous transport. Linear transport theory has been modified to take care of the geometrical effects and large mean free paths. The neoclassical theory (see Ref. [1], and references therein) is a great intellectual and practical success. Nevertheless, it is not able to resolve all the problems. As is well known [2], in many cases the strong deviation of the diffusion rate from classical predictions is due to nonlinear effects caused by (electrostatic as well as electromagnetic) fluctuations. In the past, many attempts have been made for a self-consistent theory of nonlinear transport; see Refs. [3,4], and references therein. Quasilinear theory and transport estimates based on the weak turbulence description are by far the most successful analytical approaches. However, it is well known [2] that they are of limited applicability. Strong plasma turbulence is a very complex and complicated problem. Analytical evaluations are generally too difficult, and therefore numerical simulations become more and more important. They lead to a huge data base with many hints for fundamental transport scalings.

For a better analytical understanding of anomalous transport in magnetically confined plasmas it was suggested, see Ref. [5], and references therein, to split the problem into two parts. One part deals with the development of fluctuations, and the other one considers the (passive) motion of (test) particles under the influence of the perturbations. Such separations are quite common in fluid turbulence where passive motion of scalars, vectors, particles, etc., has been investigated extensively. In plasma physics there exists an additional, qualitatively important reason to investigate particle motion in given stochastic fields. Perturbations in the magnetic field structure are more or less unavoidable because of errors in the coil arrangements of the devices. In addition, and recently that aspect became very important, additional coils are being used in tokamaks to control the particle and heat loads on the walls [6–8].

Anomalous charged particle transport is also a long-standing problem in astrophysical issues [9–13]. A variety of problems, such as low-energy cosmic ray penetration into the heliosphere, the transport of galactic cosmic rays in and out of the interstellar magnetic field, the Fermi acceleration mechanism, and so on, are on the top of the agenda. One of the new important questions added from the astrophysical point of view in the present context is, "What are the transport properties in stochastic magnetic fields without a very large mean field?" Then situations may arise where the Larmor radii are larger than the coherence length.

We shall apply the stochastic differential equation approach to the problem of transport of charged particles in a magnetic field [3,14]. In principle, there are at least two ways to do it. The motion of particles under the influence of a stochastic magnetic field, and in the presence of collisions, can be described by the acceleration-Langevin (*A*-Langevin) equation. Due to the complexity of this equation, this approach is not widely used. One can make use of the fact that in strongly magnetized plasmas, charged particles gyrate closely to the field lines. Therefore, very often [15–18] the simplified velocity-Langevin (*V*-Langevin) model is being used. The latter approximates the *A*-Langevin equation for small gyroradii. It can be derived from the *A*-Langevin equation by integration in time and applying the drift approximation. Thus, the *V*-Langevin approach assumes very large (guiding) magnetic fields such that the guiding center picture becomes meaningful. Note that the stochastic component of the magnetic field is usually weak, i.e. the stochastic fields themselves do not justify the drift approximation. When a strong confining magnetic field is additionally present, such as in tokamaks, the *V*-Langevin approach is justified. In several papers [18–20] stochastic perturbations in the presence of a strong mean field were investigated. It was found that the perturbations have a notable influence on the transport of the particles across the mean magnetic field. Two questions remain: (i) What is the finite Larmor radius correction and (ii) how does the transport looks like when no strong guiding magnetic field is present? The latter case is realized in many astrophysical situations with random magnetic fields. To answer the two questions, a procedure based on the A-Langevin equation is necessary.

In the present paper, we concentrate on the solution of the A-Langevin equation. The interaction forces are mimicked by phenomenological damping and acceleration terms. Based on the general solution of the equation of motion we calculate the velocity correlation function, leading to the diffusion tensor. Generally, the exact analytical solution of the problem is not possible, or at least extremely complicated. Nevertheless, it is possible to make some estimations in different limiting cases assuming that the random perturbations of the magnetic field are weak.

The main assumptions of the present work are the following. First, we assume static magnetic disturbances and thereby neglect the electric force on the particles. This is justified as long as the propagation velocity of the magnetic fluctuations is small compared to the typical velocity of the particles. Fast, time-dependent perturbations, which may occur due instabilities or specific experimental arrangements, are beyond the scope of the present investigation. Furthermore, we assume Gaussian Eulerian correlation functions (fulfilling the constraint $\vec{\nabla}\cdot\vec{B}=0$). The main additional assumption, known as Corrsin approximation [21,22], allows one to derive equations for the Lagrange correlator. The latter is the main ingredient for the diffusion coefficient. Recently, interesting developments to incorporate long-distance correlations into the theory appeared. For example, Ref. [12] deals with the separation of adjacent field lines in two-component turbulence consisting of a slab component that varies only along the magnetic field, as well as a two-dimensional component that varies only in the two transverse directions, which seems to be a good model for the solar wind turbulence. The nonlinear effect of magnetic line trapping on the transport of particles in stochastic magnetic fields was studied using the decorrelation trajectory method [23–25]. Compared to these papers, here we neglect the trapping of field lines and allow decorrelations of particles from the magnetic field lines due to collisions.

The paper is organized as follows. In the next section we present the mathematical formulation of the problem. The solution of the equation of motion as well as the velocity correlation function (VCF), expressed through the Lagrange correlator of the magnetic field, are presented. As multiple stochastic processes are involved, we have to average the VCF with respect to each stochastic variable. The Lagrangian correlation function (LCF) of the magnetic field is identified as the most important contribution to the VCF with regard to anomalous transport. In the Sec. III we estimate the Lagrange magnetic field correlator within the Corrsin approximation. For the latter, a criterion of validity is known in terms of the Kubo number [23–25]. Small Kubo numbers allow the Corrsin approximation, whereas Kubo numbers larger than 1 correspond to the percolation limit. The Corrsin approximation allows us to reformulate the problem in terms of a differential equation for the mean square displacement. In Sec. IV we present general results for the Lagrangian velocity correlations. The latter are the basis for the results of the corresponding diffusion constants presented in Sec. V C. The quasilinear, the subdiffusive, and the Rechester-Rosenbluth [26] regimes, respectively, are found by analytical treatments of the A-Langevin VCF. The effects of finite Larmor radii are discussed. Section V C is concluded by the discussion of the situations without dominating guiding fields. Numerical simulations, verifying our results, are presented for typical cases. Finally, in Sec. VI, a short summary and an outlook to the percolative regime concludes the paper. Some mathematical details are placed in appendixes.

II. THE A-LANGEVIN APPROACH

A. General formulation

In general, we consider a magnetic field of the form

$$\mathbf{B}=B_0(b_0\mathbf{e}_z+b_z\mathbf{e}_z+b_x\mathbf{e}_x+b_y\mathbf{e}_y), \tag{1}$$

composed of a guiding field B_0b_0 in the z direction and a perturbation. The factor B_0 takes care of the dimension of the magnetic field. We call the x and y components the perpendicular components of the perturbations $\mathbf{b}=(b_x,b_y,b_z)$. We shall subdivide our investigation into two parts. The first case corresponds to the situation with a strong guiding field $|b_0|\gg|b_z|$, $|b_x|$, $|b_y|$, causing an asymmetry between parallel and perpendicular directions. Then

$$\mathbf{B}=B_0(b_0\mathbf{e}_z+b_x\mathbf{e}_x+b_y\mathbf{e}_y), \tag{2}$$

can be used. For tokamak applications it will be appropriate to assume such a strong guiding field. Then the Larmor radii are small. It is expected that results derived from the A-Langevin equation will agree with those from the V-Langevin equation, to lowest order.

The second situation corresponds to small or vanishing guiding fields. In that case

$$\mathbf{B}=B_0(b_z\mathbf{e}_z+b_x\mathbf{e}_x+b_y\mathbf{e}_y), \tag{3}$$

should be used. We define a gyro-frequency unit $\Omega=ZeB_0/mc$. Here, m is the test particle (electron or ion) mass, and Ze is the total charge. Then the typical Larmor frequency is given by $\Omega_L=\Omega b_0$. The Larmor radius is defined as $\rho_L=v_{\mathrm{th}}/(\Omega b_0)$ with the thermal velocity v_{th}.

The A-Langevin equation is the equation of motion for a single particle, experiencing the effect of the magnetic field (including its stochastic contribution) and random collisions through $\mathbf{a}$. A friction parameter ν models the average effect of these collisions

$$\frac{d}{dt}\mathbf{u}=\frac{Ze}{mc}\mathbf{u}\times\mathbf{B}-\nu\mathbf{u}+\mathbf{a}. \tag{4}$$

Integration of $\mathbf{u}(t)$ leads to the trajectory of the particle $\mathbf{R}(t)=\int_0^t\mathbf{u}(t')dt'$. We will refer to this trajectory later.

The mathematical description of the problem is not yet complete. Assumptions on the stochastic properties of the variables are necessary. We assume

$$\langle \mathbf{a} \rangle = 0, \quad \langle a_i(t_1) a_j(t_2) \rangle = A \delta_{ij} \delta(t_1 - t_2), \tag{5}$$

$$\langle \mathbf{b} \rangle = 0, \quad \langle b_i(t_1) b_j(t_2) \rangle = \beta^2 L_{ij}. \tag{6}$$

Details of the statistics of the magnetic field will be presented in the next section.

B. Solution of the *A*-Langevin equation

When solving the A-Langevin equation, a sequence of transformations is helpful. The otherwise straightforward calculation, which is summarized in Appendix A, shows that the solution can be written in the form

$$\mathbf{u}(t) = R_3(-\Omega b_0 t) G(0,t) \mathbf{u}_0 e^{-\nu t} + R_3(-\Omega b_0 t) e^{-\nu t} \times \int_0^t G(t',t) R_3(\Omega b_0 t') \mathbf{a}(t') e^{\nu t'} dt'. \tag{7}$$

We introduced the rotational matrices $R_i(\alpha)$ of the SO(3) group and the propagation function $G(t_2,t_1)$ (Greens function)

$$G(t_2,t_1) = \mathcal{T}\left[\exp\left(-\Omega \int_{t_2}^{t_1} V(t') dt'\right)\right]. \tag{8}$$

Here, $\mathcal{T}$ is the time ordering operator. The operator V is given by

$$V(t) = \begin{pmatrix} 0 & -V_z(t) & V_y(t) \\ V_z(t) & 0 & -V_x(t) \\ -V_y(t) & V_x(t) & 0 \end{pmatrix}. \tag{9}$$

The entries are

$$V_x(t) = \cos(\Omega b_0 t) b_x(t) - \sin(\Omega b_0 t) b_y(t), \tag{10}$$

$$V_y(t) = \sin(\Omega b_0 t) b_x(t) + \cos(\Omega b_0 t) b_y(t), \tag{11}$$

$$V_z(t) = b_z(t). \tag{12}$$

A special solution exists when no perturbation is present ($\mathbf{b}=0$),

$$\mathbf{u}(t) \equiv \boldsymbol{\eta}(t) = R_3(-\Omega b_0 t) \mathbf{u}_0 e^{-\nu t} + R_3(-\Omega b_0 t) e^{-\nu t} \times \int_0^t R_3(\Omega b_0 t') \mathbf{a}(t') e^{\nu t'} dt'. \tag{13}$$

The correlation functions in the unperturbed case

$$\langle \boldsymbol{\eta}_\perp(t+\tau) \boldsymbol{\eta}_\perp(t) \rangle_\perp = \frac{v_{\text{th}}^2}{2} \exp(-\nu\tau) \cos(\Omega b_0 \tau), \tag{14}$$

$$\langle \eta_\parallel(t+\tau) \eta_\parallel(t) \rangle_\parallel = \frac{v_{\text{th}}^2}{2} \exp(-\nu\tau), \tag{15}$$

correspond to classical transport. The trajectory in the unperturbed regime is a combination of helical motion and exponential damping given by $\mathbf{R}_0(t) = \int_0^t \boldsymbol{\eta}(t') dt'$. With this definition, we shift the problem of averaging over collisions and initial velocities to the simpler task of averaging the new stochastic variables $\boldsymbol{\eta}$. Comparing Eqs. (7) and (13), we can take care of the perturbation field by using the propagation equation in the form

$$\mathbf{u}(t) \approx R_3(-\Omega b_0 t) e^{-\nu t} G(0,t) R_3(\Omega b_0 t) \boldsymbol{\eta}(t). \tag{16}$$

This procedure is in agreement with that for the V-Langevin equation. It still contains the effects of particle gyration. We can further simplify for $\beta \ll 1$, which allows us to expand the propagator in the form $G(0,t) \approx I + \int_0^t V(t') dt'$. The final result is

$$\mathbf{u}(t) = \boldsymbol{\eta}(t) + \int_0^t R_3(-\Omega b_0 t) e^{-\nu t} V(t') R_3(\Omega b_0 t) \boldsymbol{\eta}(t) dt'. \tag{17}$$

C. The velocity correlation function

The transport of particles can be deduced from the velocity correlation function. Once the Lagrangian velocity correlation is known, the mean square displacement (MSD) and the diffusion coefficient are typically obtained from the Green-Kubo formula

$$\frac{d^2}{dt^2} \langle \delta r_i^2 \rangle = 2 \frac{d}{dt} D(t) = \langle \langle \langle u_i(t_1) u_i(t_2) \rangle_{\mathbf{b}} \rangle_\perp \rangle_\parallel, \quad i = x,y,z. \tag{18}$$

The Green-Kubo formula connects the MSD, respectively the running diffusion coefficient $D(t)$, with the velocity correlation function. It should be solved with the initial conditions $D(0)=0$ and $\langle \delta r_i^2(0) \rangle = 0$. The solution (17) can be used to construct the perpendicular velocity correlation

$$u_x(t_1) u_x(t_2) = \eta_x(t_1) \eta_x(t_2) + \Omega^2 \eta_z(t_1) \eta_z(t_2) \left\{ \int_0^{t_1} \sin[\Omega b_0(\tau_1 - t_1)] b_x(\tau_1) + \cos[\Omega b_0(\tau_1 - t_1)] b_y(\tau_1) d\tau_1 \int_0^{t_2} \sin[\Omega b_0(\tau_2 - t_2)] b_x(\tau_2) + \cos[\Omega b_0(\tau_2 - t_2)] b_y(\tau_2) d\tau_2 \right\} + \Omega^2 \eta_y(t_1) \eta_y(t_2) \int_0^{t_1} \int_0^{t_2} b_z(\tau_1) b_z(\tau_2) d\tau_1 d\tau_2. \tag{19}$$

The influences of the magnetic perturbations appear explicitly. We shall call the contribution from the perturbation terms the anomalous contribution, thereby distinguishing between classical transport, described by Eq. (14), and the anomalous transport. We write

$$u_x(t_1)u_x(t_2) = [\eta_x(t_1)\eta_x(t_2)]^{\mathrm{CL}} + [u_x(t_1)u_x(t_2)]^{\mathrm{AN}}. \quad (20)$$

The parallel velocity correlation is formulated in a similar manner,

$$\begin{aligned} u_z(t_1)u_z(t_2) = \eta_z(t_1)\eta_z(t_2) + \Omega^2[\eta_x(t_1)\eta_x(t_2) + \eta_y(t_1)\eta_y(t_2)] \\ \times \int_0^{t_1}\int_0^{t_2} \cos \Omega b_0[t_1 - \tau_1 - (t_2 - \tau_2)] \\ \times b_\perp(\tau_1)b_\perp(\tau_2)d\tau_1 d\tau_2. \end{aligned} \quad (21)$$

Note the important fact that the two expressions (19) and (21) coincide for $b_0 \to 0$. Without the guiding field, there is no preferred direction, and the transport coefficients for parallel and perpendicular transport are evidently identical. In Ref. [9] this tendency was observed numerically. It is an essential advantage of the A-Langevin approach to include the limiting case consistently. Any method based on the guiding center assumption fails to describe the transition.

D. Strong guiding fields

The velocity correlation functions still require averaging with respect to the stochastic variables. Now, we restrict our analysis to the case of (relatively) strong guiding fields, $b_0 \gg 1$. This corresponds to a magnetic field configuration described by Eq. (2). In such a case, we can neglect the effect of the perturbations in the z direction and apply the approximations (5) and (6) developed in Appendix B. The result is

$$\begin{aligned} \langle u_x(t_1)u_x(t_2)\rangle^{\mathrm{AN}}_{\mathbf{b},\perp,\parallel} &= \frac{1}{b_0^2}\langle \eta_z(t_1)\eta_z(t_2)\langle\langle b_x(t_1)b_x(t_2)\rangle_{\mathbf{b}}\rangle_\perp\rangle_\parallel \\ &+ \frac{1}{\Omega^2 b_0^4}\langle \eta_z(t_1)\eta_z(t_2)\langle\langle b_y'(t_1)b_y'(t_2)\rangle_{\mathbf{b}}\rangle_\perp\rangle_\parallel \\ &\equiv L^{(0)} + L^{(1)}. \end{aligned} \quad (22)$$

A specific order of averages occurs. The average of the parallel collisions, covered by η_z, is especially involved since we must also include all dependencies on η_z remaining in the perturbation fields $\mathbf{b}$. The first term on the right-hand side of Eq. (22) $L^{(0)}$, does not include any effects of the finite Larmor radii. It describes the dynamics similar to the V-Langevin equations from a pure guiding center perspective. First order Larmor radius effects are included in the second term $L^{(1)}$ (and all higher order corrections can also be found by the method sketched in Appendix B). At this stage we are left with the problem to find appropriate expressions for the Lagrangian b-field correlations, respectively the correlations for the derivations of the b fields.

III. MAGNETIC FIELD CORRELATIONS

A. Eulerian correlation function

In this section we still assume strong guiding fields. Appropriate estimates for the Lagrangian correlation functions have been intensively discussed in literature. Common approaches start with the spatial Eulerian correlation function for the two-dimensional perturbations, in Gaussian form

$$\begin{aligned} \langle \mathbf{b}(r)\mathbf{b}(0)\rangle &\equiv E(\mathbf{r}) \\ &= \begin{pmatrix} 1 - y/\lambda_\perp^2 & 0 \\ 0 & 1 - x/\lambda_\perp^2 \end{pmatrix} \exp\left(-\frac{x^2+y^2}{\lambda_\perp^2} - \frac{z^2}{\lambda_\parallel^2}\right). \end{aligned} \quad (23)$$

Two important length scales define the stochastic magnetic field fluctuations, namely, the correlation lengths $\lambda_\parallel$ and $\lambda_\perp$. Note that we shall assume finite parallel correlation lengths for the $\mathbf{b}$ perturbations such that a slab model [12] is not covered by the following treatment.

It is convenient to introduce also the Fourier transform of $\mathbf{b}(\mathbf{R})$,

$$\mathbf{b}(\mathbf{R}(t)) = \int \mathbf{b}(\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{R}(t))d\mathbf{k}. \quad (24)$$

The correlation spectrum of Eq. (23) is then given by

$$\tilde{E}(\mathbf{k}) = (\mathbf{k}_\perp^2\delta_{ij} - k_i k_j)A(\mathbf{k}) \quad (25)$$

with

$$A(\mathbf{k}) = (2\pi)^{-3/2}\lambda_\parallel\lambda_\perp^4\beta^2\exp\left(-\frac{1}{2}\lambda_\parallel^2 k_\parallel^2 - \frac{1}{2}\lambda_\perp^2\mathbf{k}_\perp^2\right). \quad (26)$$

B. The Corrsin approximation

The widely adopted approximation method due to Corrsin [21] assumes that the correlation function and the trajectory can be averaged independently. Details of this procedure can be found in Ref. [22]. Within the Corrsin approximation, the Lagrangian correlator is calculated via the integral

$$\langle b_{x,y}(t)b_{x,y}(0)\rangle_b = \int_{-\infty}^{\infty} E(\mathbf{r})\langle\delta(\mathbf{r} - \mathbf{R}(t))\rangle_{\mathbf{b}} d\mathbf{r}. \quad (27)$$

The averaged propagator $\langle\delta(\mathbf{r} - \delta\mathbf{r}(t))\rangle$ can be calculated by using the Fourier expression of the δ distribution and the Fourier transform of $E(\mathbf{r})$,

$$\langle b_{x,y}(t)b_{x,y}(0)\rangle_b = \int_{-\infty}^{\infty} \tilde{E}(\mathbf{k})\langle\exp(-i\mathbf{k}\cdot\mathbf{R}(t))\rangle_{\mathbf{b}} d\mathbf{k}. \quad (28)$$

The trajectory $\mathbf{R}(t)$ has to be inserted into the Corrsin approximation. Since the trajectory depends on $\boldsymbol{\eta}$, it has to be included in the averaging procedures regarding parallel and perpendicular motion.

The second term on the right-hand side of Eq. (22) requires the correlation of the derivatives of $\mathbf{b}$. For that we differentiate Eq. (24) in time,

$$\mathbf{b}'(\mathbf{R}(t)) = -i\int \mathbf{k}\cdot\mathbf{R}'(t)\mathbf{b}(\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{R}(t))d\mathbf{k}, \quad (29)$$

which directly leads to the expression

$$\langle b'_{x,y}(t)b'_{x,y}(0)\rangle_b = -\int_{-\infty}^{\infty}(\mathbf{k}\cdot\mathbf{R}'(t))^2\tilde{E}(\mathbf{k}) \times\langle\exp(-i\mathbf{k}\cdot\mathbf{R}(t))\rangle_{\mathbf{b}}d\mathbf{k}. \quad (30)$$

Equation (27) is restricted to a certain domain of validity, defined in terms of the Kubo number

$$K = \frac{\beta\lambda_\parallel}{b_0\lambda_\perp}. \quad (31)$$

The Kubo number is generally defined as the ratio of the distance a particle travels during an autocorrelation time over the correlation distance. Large Kubo numbers lead to a failure of the independence hypothesis, and in this case Eq. (27) is not applicable. Some recent works [23–25] presented suitable replacements for the Corrsin approximation which are valid for larger Kubo numbers. As a matter of fact, the decorrelation trajectory method (DCT) is quite involved and its application within the A-Langevin framework will be presented in a separate work. For magnetic fluctuations obeying $K<1$, the Corrsin approximation remains valid.

IV. LAGRANGIAN VELOCITY CORRELATIONS

A. The guiding center limit

Strong guiding fields, $b_0 \gg 1$ reduce the Larmor radius of the gyration around the field lines. Obviously, to zeroth order, the position of a particle can be approximated by its guiding center, and the influences of fluctuations aligned with the mean field can be neglected, i.e., $b_z \approx 0$. Anomalous transport is then dominantly described by the first term on the right-hand side of Eq. (22).

Let us further define two functions determining the classical transport behavior, namely,

$$\varphi_{\perp,\parallel}(\tau) = \int_{t_2}^{t_1}\langle\eta_{\perp,\parallel}(t_1)\eta_{\perp,\parallel}(t')\rangle dt' = \chi_{\perp,\parallel}(1-e^{-\nu\tau}) \quad (32)$$

and

$$\psi_{\perp,\parallel}(\tau) = \int_{t_2}^{t_1}\int_{t_2}^{t_1}\langle\eta_{\perp,\parallel}(t')\eta_{\perp,\parallel}(t'')\rangle dt'dt'' = \frac{2\chi_{\perp,\parallel}}{\nu}(\nu\tau-1+e^{-\nu\tau}). \quad (33)$$

Here, the classical expressions

$$\chi_\perp = \frac{v_{\mathrm{th}}^2\nu}{2(\Omega b_0)^2}, \quad \chi_\parallel = \frac{v_{\mathrm{th}}^2}{2\nu} \quad (34)$$

have been used. The function $\psi_{\perp,\parallel} = \langle\delta r_i^2(t)\rangle$ represents the mean square displacement in the classical case. For

$$L^{(0)} = \frac{1}{b_0^2}\langle\eta_z(t_1)\eta_z(t_2)\langle b_\perp(t_1)b_\perp(t_2)\rangle_b\rangle_\parallel. \quad (35)$$

we apply the Corrsin approximation, transforming the correlation function into the Lagrangian frame of reference

$$L^{(0)} = \frac{1}{b_0^2}\int_{-\infty}^{\infty}\tilde{E}(\mathbf{k})\langle\eta_z(t_1)\eta_z(t_2)\langle\exp(-i\mathbf{k}\cdot\mathbf{R}(t))\rangle_{\mathbf{b}}\rangle_{\perp\parallel}d\mathbf{k}. \quad (36)$$

The perpendicular average is calculated using the cumulant expansion with the result

$$L^{(0)} = \frac{1}{b_0^2}\int_{-\infty}^{\infty}\tilde{E}(\mathbf{k}) \times\left\langle\eta_z(t+\tau)\eta_z(t)\exp\left(-ik_z\int_t^{t+\tau}\eta_z(\tau')d\tau'\right)\right\rangle_\parallel \times\exp\left(-\frac{1}{2}k_x^2\langle\delta x^2\rangle-\frac{1}{2}k_y^2\langle\delta y^2\rangle\right)d\mathbf{k}. \quad (37)$$

Note that the mean square displacements $\langle\delta x^2\rangle$ and $\langle\delta y^2\rangle$ still contain the anomalous parts and should not be confused with the classical $\psi_{\perp,\parallel}$ terms. The combined average of the parallel motion is performed using the prescription shown in Appendix C. We obtain

$$L^{(0)} = \frac{1}{b_0^2}\int_{-\infty}^{\infty}\tilde{E}(\mathbf{k})\left[\frac{v_{\mathrm{th}}^2}{2}e^{-\nu t}-k_z^2\varphi_\parallel^2\right]\exp\left(-\frac{1}{2}k_z^2\psi_\parallel\right) \times\exp\left(-\frac{1}{2}k_x^2\langle\delta x^2\rangle-\frac{1}{2}k_y^2\langle\delta y^2\rangle\right)d\mathbf{k}. \quad (38)$$

Inserting Eq. (25), and performing the integration over $\mathbf{k}$, we finally find the Lagrangian velocity correlation function of the guiding center motion

$$L^{(0)} = \frac{\beta^2}{b_0^2}\left[\frac{v_{\mathrm{th}}^2}{2}e^{-\nu\tau}-\frac{\varphi_\parallel^2}{\lambda_\parallel^2}\frac{1}{\left(1+\frac{\psi_\parallel}{\lambda_\parallel^2}\right)}\right] \times\frac{1}{\left(1+\frac{\langle\delta x^2\rangle}{\lambda_\perp}\right)^2\left(1+\frac{\psi_\parallel}{\lambda_\parallel^2}\right)^{1/2}}. \quad (39)$$

In the last step we used the symmetry of the system $\langle\delta x^2\rangle = \langle\delta y^2\rangle$.

B. Finite Larmor radius corrections

In the case of strong guiding fields, the main advantage of the A-Langevin approach is the capability to calculate finite Larmor radius corrections. As long as the guiding center approximation can be applied, the system is determined by $L^{(0)}$. For smaller guiding fields, the gyration around the field lines contributes to the transport, and finite values of the Larmor radius must be taken into account. Depending on the ratio β/b_0 of fluctuations and guiding field, the finite Larmor radius effects can become important. For $b_0 \gg \beta$, the Larmor radius is identified by $\rho_L = v_{\rm th}/(\Omega b_0)$. Finite Larmor radii change the transport behavior and appear in Eq. (22) via the additional (first order) perturbation term $L^{(1)}$. We will still assume that the guiding field is predominantly stronger than the fluctuations. The correction turns out to be

$$L^{(1)} = -\frac{1}{\Omega^2 b_0^4}\int\left\{\tilde{E}(\mathbf{k})\left[\left\langle (k_x^2\eta_x(t_1)\eta_x(t_2)+k_y^2\eta_y(t_1)\eta_y(t_2))\exp\left(-ik_x\int_{t_2}^{t_1}\eta_x(t')dt' - ik_y\int_{t_2}^{t_1}\eta_y(t')dt'\right)\right\rangle_\perp \times\left\langle \eta_z(t_1)\eta_z(t_2)\exp\left(-ik_z\int_{t_2}^{t_1}\eta_z(t')dt'\right)\right\rangle_\parallel + k_z^2\left\langle \eta_z^2(t_1)\eta_z^2(t_2)\exp\left(-ik_z\int_{t_2}^{t_1}\eta_z(t')dt'\right)\right\rangle_\parallel \times\left\langle \exp\left(-ik_x\int_{t_2}^{t_1}\eta_x(t')dt' - ik_y\int_{t_2}^{t_1}\eta_y(t')dt'\right)\right\rangle_\perp\right]\right\}d\mathbf{k}. \tag{40}$$

The last term on the right-hand side can be approximated by

$$k_z^2\left\langle \eta_z^2(t_1)\eta_z^2(t_2)\exp\left(-ik_z\int_{t_2}^{t_1}\eta_z(t')dt'\right)\right\rangle_\parallel \approx v_{\rm th}^4 k_z^2\exp\left(-\frac{1}{2}k_z^2\psi_\parallel\right). \tag{41}$$

This approximation has been checked *a posteriori*. Additionally, we used the fact that the quadratic η_z terms obey a long tail correlation which can be estimated by $\langle \eta_z^2(t_1)\eta_z^2(t_2)\rangle_\parallel = v_{\rm th}^4$.

The other essential steps are similar to those of the previous section. Finally, the finite Larmor radius correction is given by

$$L^{(1)} = -\frac{\rho_L^2}{v_{\rm th}^2}L^{(0)}_{\lambda_\perp\to\infty}\left(\frac{4\chi_\perp \nu e^{-\nu t}}{\lambda_\perp^2\left(1+\frac{\psi_\perp}{\lambda_\perp^2}\right)^3} - \frac{18\varphi_\perp^2}{\lambda_\perp^4\left(1+\frac{\psi_\perp}{\lambda_\perp^2}\right)^3}\right) - \frac{\rho_L^2\beta^2}{b_0^2}\frac{v_{\rm th}^2}{4\lambda_\parallel\left(1+\frac{\psi_\perp}{\lambda_\perp^2}\right)^2\left(1+\frac{\psi_\parallel}{\lambda_\parallel^2}\right)^{3/2}}. \tag{42}$$

Here, $L^{(0)}_{\lambda_\perp\to\infty}$ means Eq. (39) in the limit $\lambda_\perp\to\infty$, and $L^{(1)}$ is a correction term affecting each regime stated in the previous section. For small Larmor radii the influence of this correction vanishes.

V. DIFFUSION COEFFICIENTS

A. Guiding center results

Equation (39) can be introduced into Eq. (18) yielding a differential equation for the perpendicular mean square displacement and the running diffusion coefficient. The result is equivalent to the correlation functions found in Refs. [15,18] (V-Langevin equation plus Corrsin approximation) and [17] (V-Langevin equation plus MDIA approximation). The latter treatments are restricted to the guiding center approximation. It is natural that the corresponding result appears here as the zeroth order term in a perturbation series in terms of the Larmor radius. Thus, we have reproduced (in the strong mean field limit) the well-known diffusion regimes in the guiding center approximation. We will not go into the details and only summarize the important regimes with the corresponding references.

1. The quasilinear regime

The quasilinear regime refers to a domain in which collisions are absent, $\nu=0$, and the perpendicular correlation length tends to infinity, $\lambda_\perp\to\infty$. No implicit dependence on the perpendicular MSD remains. The functions $\varphi_\parallel$ and $\psi_\parallel$ can be expanded in power series at $\nu=0$, yielding $\varphi_\parallel=(v_{\rm th}^2/2)t$ and $\psi_\parallel=(v_{\rm th}^2/2)t^2$. Obviously, only a ballistic motion along the field lines prevails. Equation (18) is solved by straightforward integration and leads to the quasilinear diffusion coefficient ($t\to\infty$),

$$D_{ql}^{(0)} \approx \frac{1}{\sqrt{2}}v_{\rm th}\frac{\beta^2}{b_0^2}\lambda_\parallel. \tag{43}$$

2. The subdiffusive regime

Contrary to the quasilinear regime, we allow here for collisions of the particles *along* the field lines, but still keep the condition $\lambda_\perp\to\infty$. Of course, the first assumption (on the collisions) is somehow artificial. The transport in the z direction is now diffusive and the anomalous part of the perpendicular transport becomes subdiffusive [15].

3. The Kadomtsev-Pogutse regime

Two diffusion scalings are often referred to as Kadomtsev-Pogutse [27] limit. The first one, which we will refer to as percolation limit, is characterized by an infinite parallel correlation length $\lambda_\parallel\to\infty$. Integration of Eq. (18) leads to a diffusion coefficient $D\approx(\beta/b_0)\lambda_\perp v_{\rm th}$. In a series of papers [23–25] it was shown that this prediction is not correct. The problem is caused by the failure of the Corrsin approximation for high Kubo numbers. By the present restriction to small Kubo numbers we exclude this case.

If the decorrelation of the particles is caused by classical collisional events only, the anomalous terms in Eq. (39) can be neglected and $\langle \delta x^2(t)\rangle = \psi_\perp$ should be used. Furthermore it is assumed that collisional effects are more important than the changes due to the magnetic field perturbations. This leads to the second diffusion scaling known as Kadomtsev-Pogutse limit [12]

$$D_{\mathrm{KP}}^{(0)} \approx \frac{3}{4}\pi\frac{\beta^2}{b_0^2}\frac{\lambda_\parallel}{\lambda_\perp}\sqrt{\chi_\perp \chi_\parallel}. \tag{44}$$

Note that here collisions cause decorrelations from the field lines.

4. The Rechester-Rosenbluth regime

A situation with finite correlation lengths, strong collisionality, and larger β than in the Kadomtsev-Pogutse regime, corresponds to the Rechester-Rosenbluth regime [26]. Using approximations for short and long times, the correlation function (39) can be shown to reflect the characteristic scaling of this regime, as was pointed out in Ref. [17]. We have

$$D_{\mathrm{RR}}^{(0)} \approx \frac{2\lambda_\perp^2}{\pi L_k^2}\frac{\chi_\parallel}{\ln^2\left(2\sqrt{\frac{2}{\pi}\frac{\chi_\parallel \lambda_\perp^2}{\chi_\perp L_K^2}}\right)}, \tag{45}$$

where

$$L_K \approx \sqrt{\frac{2}{\pi}\frac{b_0^2\lambda_\perp^2}{4\beta^2\lambda_\parallel}} \tag{46}$$

is the Kolmogorov length.

B. Finite Larmor radius corrections

1. The Rechester-Rosenbluth regime

We start with parameters in the Rechester-Rosenbluth regime. Similar dependencies due to finite Larmor radii occur in the other parameter regimes, as will be discussed later. First, we demonstrate the modification of the diffusive behavior due to finite Larmor radii by showing the time dependence of the MSD. Together with the zeroth order terms, the correlation function has to be introduced into the Green-Kubo formula to calculate the MSD. Its evaluation was done numerically. Typical results are shown in Fig. 1. Generally, the increase of ρ_L leads to a reduction of the diffusion coefficient. We varied the Larmor radius by reducing the mean field b_0. In the uncorrected $L^{(0)}$ case, for the chosen parameters the scaling of Eq. (45) holds and the diffusion increases with decreasing field strength b_0. The first order term $L^{(1)}$ causes a significantreduction of the transport.

Figure 1 clearly shows that the running diffusion coefficient $D(t)$ converges to a constant value (here and in the following called *the* diffusion coefficient). The dependencies

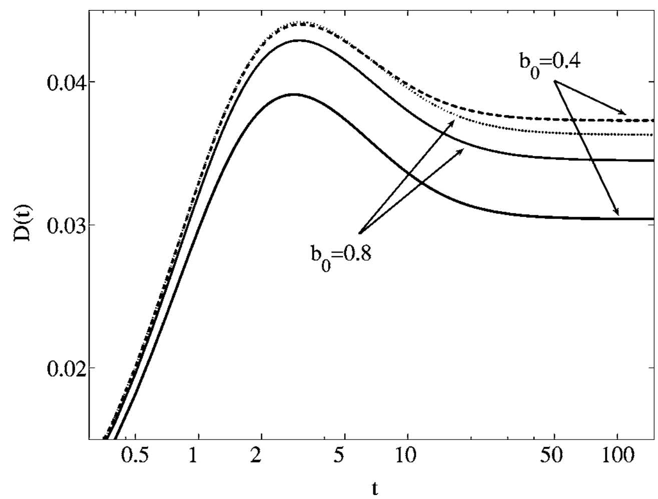

FIG. 1. Solution of the Green-Kubo equation with the correlation function $L^{(0)}$ only (dashed lines) and $L^{(0)}+L^{(1)}$ (solid lines) for two different values of b_0 (strong guiding field), respectively. Shown is the diffusion coefficient D (unit v_{th}^2/Ω) versus time t (unit Ω^{-1}). The parameters are $\lambda_\parallel=\lambda_\perp=5.87[v_{\mathrm{th}}/\Omega]$, $\epsilon=\beta/b_0=0.3$, and $\nu/\Omega=0.2$.

of the diffusion coefficient (with Larmor radius corrections in the Rechester-Rosenbluth regime) on physical parameters such as temperature, fluctuation strength, etc., is shown in Figs. 2 and 3. Figure 2 shows a quadratic decrease of the diffusion coefficient with increasing strength of the Larmor radius. Obviously, the Larmor radius depends on temperature and magnetic field strength. In Fig. 3 one first recognizes the strong dependence of the diffusion coefficient on the Kubo number $K\sim\beta$, known from the original Rechester-Rosenbluth scaling [26] with the perturbation strength β. For all Kubo numbers ($\leqslant 1$) the diffusion is reduced due to finite Larmor radii.

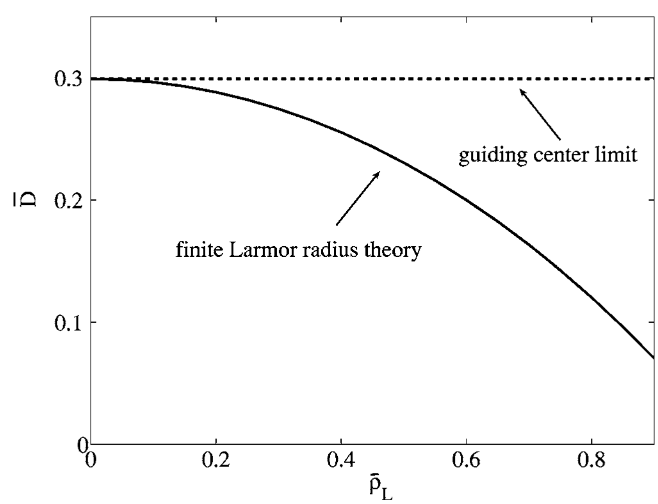

FIG. 2. Effect of finite Larmor radius corrections on the diffusion coefficient in dependence of the Larmor radius, calculated in the Rechester-Rosenbluth parameter regime. Shown is the normalized diffusion coefficient $\bar{D}=D/(\nu\lambda_\perp^2)$ versus the normalized Larmor radius $\bar{\rho}_L=\rho_L/\lambda_\parallel$ for $K=0.4$.

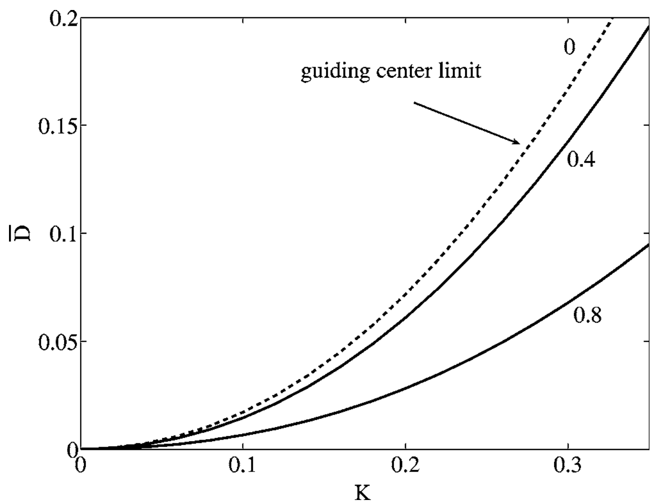

FIG. 3. Effect of finite Larmor radius corrections on the diffusion coefficient in dependence of the Kubo number K, calculated in the Rechester-Rosenbluth parameter regime. Shown is the normalized diffusion coefficient $\bar{D}=D/(\nu\lambda_\perp^2)$ versus K for the three values 0, 0.4, 0.8, respectively, of the normalized Larmor radius $\bar{\rho}_L=\rho_L/\lambda_\parallel$.

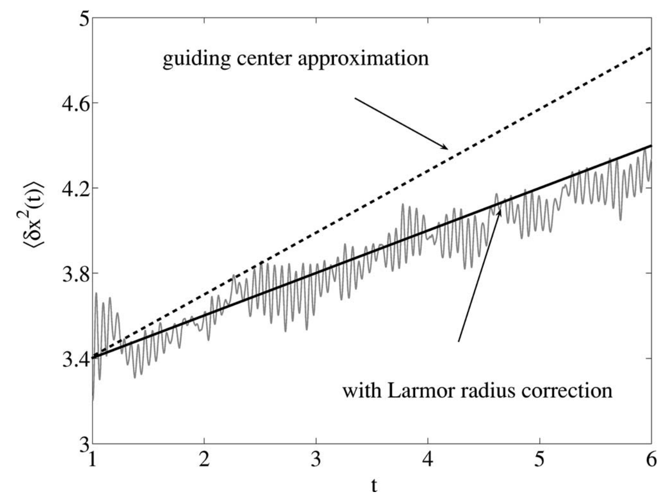

FIG. 4. Numerical simulation of the A-Langevin equation and comparison with the analytical predictions, for the case of a strong guiding field. Shown is the mean square displacement $\langle(\delta x)^2\rangle$ (unit $v_{\rm th}^2/\Omega^2$) versus time t (unit $100\ \Omega^{-1}$). The parameters are $\lambda_\parallel=2\lambda_\perp=2[v_{\rm th}/\Omega]$, $\epsilon=\beta/b_0=0.4$, $b_0=1.5$, and $\nu/\Omega=0.05$.

2. The Kadomtsev-Pogutse regime

In the weakly anomalous regime, the finite Larmor radius corrections to the zeroth order result (44) can be calculated analytically. After some straightforward integrations we obtain from the Lagrange correlation (42)

$$D_{\rm KP}^{(1)} \approx -\frac{3\pi}{2}\rho_L^2 K^2 \frac{\lambda_\perp}{\lambda_\parallel}\sqrt{\frac{\chi_\perp}{\chi_\parallel}}\nu. \qquad (47)$$

As before, an increase of ρ_L leads to a reduction of the diffusion coefficient. The analytical formula (47) clearly shows, in addition, the dependencies on the correlation lengths, the collision frequency, and the perturbation strength.

3. The quasilinear regime

Also in the quasilinear regime, the finite Larmor radius corrections to the zeroth order result (43) can be calculated analytically. After some straightforward integrations we obtain from the Lagrange correlation (42)

$$D_{ql}^{(1)} \approx -\frac{1}{4\sqrt{2}} v_{\rm th}\rho_L^2 \frac{\beta^2}{b_0^2}\frac{1}{\lambda_\parallel}. \qquad (48)$$

Again, an increase of ρ_L leads to a reduction of the diffusion coefficient. It is interesting to note that in Eq. (48) the dependence on $\lambda_\parallel$ is opposite to the zeroth order result (43).

4. Comparison with numerical simulations

In order to independently check the analytical results, we performed numerical simulations of the A-Langevin equation. The numerical procedure is based on the Monte Carlo principle. Equation (4) is solved with a standard Runge-Kutta method. A white noise process models the collisions. The magnetic fluctuations are realized by a random number generator which picks correlated random numbers from the Eulerian correlation function. We propagate an ensemble of 40 to 100 particles through the stochastic environment and measure the MSD. We find excellent agreement with the analytical predictions presented in this paper. The numerical analysis was performed in parameter regimes where the influences of finite Larmor-radii occur. Figure 4 shows the simulation results for the perpendicular MSD within the Rechester-Rosenbluth regime (strong collisionality and finite $\lambda_\parallel,\lambda_\perp$). The guiding center prediction and the influence of the correction terms are also shown. The strength of the mean field was decreased to a value where effects of the correction terms are pronounced. Of course, for stronger guiding fields the difference between the guiding center predictions and the present theory will decrease. The comparison leads to an excellent verification of our theoretical prediction. Minor deviations of the simulation from the first-order theory are due to higher order Larmor radius effects. The concordance of the Monte Carlo solutions and the analytical results is also a solid confirmation of the accuracy of the Corrsin approximation for small Kubo numbers.

C. Vanishing guiding fields

So far we described the transport of particles in the presence of strong guiding fields and additional stochastic perturbations. The situation changes considerably when the mean (guiding) field is no longer present. The total B field is then given by Eq. (3), and the particle transport takes place in a dominantly stochastic environment. Some remarks on this case were made already in the appendix of Ref. [9]. We now present predictions for the collisional case.

1. Analytical estimates

The solution (16) of the A-Langevin equation simplifies to

$$\mathbf{u}(t) = G(0,t)\,\boldsymbol{\eta}(t). \tag{49}$$

We assume that the averages over the collisions (we cannot distinguish any directions) and the b field can now be applied independently, yielding

$$\langle u(t)u(0)\rangle = \frac{v_{\mathrm{th}}^2}{2}e^{-\nu t}\langle G(0,t)\rangle. \tag{50}$$

Using the cumulant expansion and the properties of the matrix V, we find the asymptotic expression

$$\langle G(0,t)\rangle = \exp(-2\Omega^2\gamma t), \tag{51}$$

with

$$\gamma = \int_0^\infty \langle b(\tau)b(0)\rangle d\tau. \tag{52}$$

The correlation function of the magnetic perturbation field will be calculated within the Corrsin approximation. The algebra is similar to that presented in the previous section. We get

$$\gamma = \int_0^\infty \int_{k>2\pi/\rho_L} \tilde{E}(k)\exp\left(-\frac{1}{2}k^2\chi\tau\right)dk\,d\tau. \tag{53}$$

Essential for the calculation is the heuristic limitation [9] of the effective integration region, namely, that the Larmor radius of the particles has to be larger than the wavelength of the modes in Eq. (53). Because the particles follow the field lines when their Larmor radius is smaller than the wavelength of the modes, we consider only the modes with $k>2\pi/\rho_L$.

The collisional diffusion coefficient (in each direction) is given by $\chi = v_{\mathrm{th}}^2/2\nu$. In Eq. (53) we use the one-dimensional perturbation spectrum

$$\tilde{E}(k) = \sqrt{2\pi}\lambda\,\exp\left(-\frac{1}{2}k^2\lambda^2\right). \tag{54}$$

Here, $\lambda = \lambda_\parallel = \lambda_\perp$. The k integration leads to

$$\gamma = \int_0^\infty \beta^2\lambda\frac{\mathrm{erfc}(\sqrt{2}\pi\sqrt{\lambda^2+\chi\tau}/\rho_L)}{\sqrt{\lambda^2+\chi\tau}}d\tau. \tag{55}$$

The integral can be performed when the approximation $\mathrm{erfc}(x)/x \approx e^{-x^2}/x$ is used,

$$\gamma \approx \frac{\beta^2\lambda\rho_L}{\sqrt{2\pi}\chi}\mathrm{erfc}\left(\sqrt{2}\pi\frac{\lambda}{\rho_L}\right). \tag{56}$$

Finally, we substitute the velocity correlator into the Green-Kubo formula and obtain for the MSD

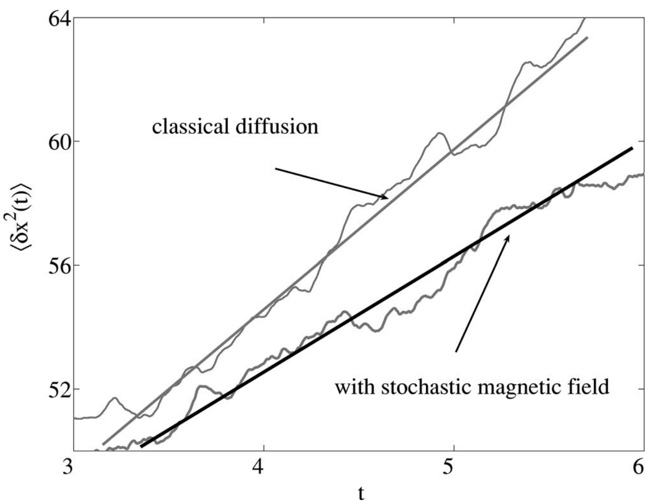

FIG. 5. Numerical simulation, with and without stochastic magnetic fields, respectively, in the case of no guiding field. Straight lines indicate the analytical predictions. Shown is the mean square displacement $\langle(\delta x)^2\rangle$ (unit $v_{\mathrm{th}}^2/\Omega^2$) versus time t (unit 100 Ω^{-1}). The parameters are $\lambda=0.1[v_{\mathrm{th}}/\Omega]$, $\beta=0$, and $\beta=0.9$, respectively, $b_0=0$ and $\nu/\Omega=0.2$.

$$\langle \delta x^2\rangle = \frac{v_{\mathrm{th}}^2}{\nu+2\Omega^2\gamma}t. \tag{57}$$

The perturbation fields acts as an effective friction. Strong, uncorrelated magnetic fluctuations will reduce the diffusion in the same way as the collisions. Note that γ vanishes for $\lambda>\rho$. In that case the fluctuations have long-range correlations, and the magnetic field does not change significantly over a certain distance.

2. Comparison with numerical simulations

The analytical prediction will now be compared with numerical simulations. Figure 5 shows simulations in the limit of vanishing guiding fields. Typically, the influence of magnetic fluctuations is then very small; one has to find the extremum of Eq. (56). We set up a suitable parameter regime to make the effect of the perturbation field visible. As can be seen from the figure, the stochastic field acts like an additional friction and reduces the gradient of the MSD. Again we found very good agreement with the analytical predictions. The latter are shown by the straight lines.

VI. CONCLUSIONS AND OUTLOOK

In this paper, on the basis of the A-Langevin equation, we have investigated finite Larmor radius effects for the diffusion of test particles in the presence of stochastic magnetic fields. For very strong guiding (mean) magnetic fields, to lowest order we recover the guiding center result which has been derived previously on the basis of the V-Langevin equation. Finite Larmor radii reduce the transport, compared to the guiding center prediction. When no strong mean magnetic field is present, the A-Langevin equation allows one to calculate the diffusion under the influence of magnetic perturbations. In the collisional case, one obtains only small deviations from the classical result. Analytical theory is in excellent agreement with numerical simulations.

The main assumption of the present theory is the Corrsin approximation. The latter requires the smallness of the Kubo number K. For $K<1$, numerical calculations support the Corrsin approximation. We also made first calculations in the percolation limit $K>1$. Preliminary results indicate a reduction of the diffusion coefficient below the value predicted by the Corrsin approximation. Details on the percolative transport and a suitable analytical approach based on the A-Langevin framework will be presented in a separate paper.

ACKNOWLEDGMENTS

This work was performed under the auspices of the SFB 591. Interesting discussions with Dmitrij Lesnik are gratefully acknowledged.

APPENDIX A: FORMAL SOLUTION OF THE A-LANGEVIN EQUATION

When solving the A-Langevin equation, we introduce the rotational matrices

$$R_i(t) = e^{l_i t}, \quad i = 1,2,3, \tag{A1}$$

and use the generators of the SO(3) group,

$$l_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad l_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$l_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A2}$$

Defining $\mathbf{l}=(l_1,l_2,l_3)$ and introducing

$$\tilde{\mathbf{u}}(t) = e^{-\nu t} R_3(-\Omega_0 b_0 t)\mathbf{u}(t), \tag{A3}$$

$$\tilde{\mathbf{a}}(t) = e^{\nu t} R_3(\Omega_0 b_0 t)\mathbf{a}(t), \tag{A4}$$

$$V(t) = R_3(\Omega_0 b_0 t)\mathbf{b}(t)\mathbf{l}\, R_3(-\Omega_0 b_0 t), \tag{A5}$$

Eq. (4) can be written as

$$\frac{d}{dt}\tilde{\mathbf{u}}(t) = V(t)\tilde{\mathbf{u}}(t) + \tilde{\mathbf{a}}(t). \tag{A6}$$

This leads to the solution

$$\tilde{\mathbf{u}}(t) = G(0,t)\tilde{\mathbf{u}}_0 + \int_0^t G(t',t)\tilde{\mathbf{a}}(t')dt'. \tag{A7}$$

The latter is given in terms of the Greens function $G(t_2,t_1)$ [28]

$$G(t_2,t_1) = \mathcal{T}\left[\exp\left(-\Omega_0\int_{t_2}^{t_1} V(t')dt'\right)\right], \tag{A8}$$

where $\mathcal{T}$ is the time ordering operator. Evaluating the operator in the integrand, we find Eqs. (9)–(12) used in the main text

APPENDIX B: MULTIPLE SCALES IN EVALUATING INTEGRALS

The expressions (19) and (21) contain, after integration, products of trigonometric functions and the magnetic perturbation fields in the integrands, e.g.,

$$I_1(t) = \int_0^t \cos[T^{-1}(t-t')]b_y(t')dt', \tag{B1}$$

$$I_2(t) = \int_0^t \sin[T^{-1}(t-t')]b_x(t')dt'. \tag{B2}$$

Here, $T=(\Omega b_0)^{-1}$, and we introduce the characteristic time τ for the b variations. In the case of strong guiding fields, $\varepsilon \equiv T/\tau \ll 1$. The integrals can be evaluated systematically by using a multiple scale perturbation method, which leads to

$$I_1(t) \approx Tb_y(t)\sin(T^{-1}t) - T^2 b_y'(t) - T^2 b_y'(t)\cos(T^{-1}t) + O(\varepsilon^2), \tag{B3}$$

$$I_2(t) \approx -Tb_x(t)\cos(T^{-1}t) - Tb_x(t) - T^2 b_x'(t)\sin(T^{-1}t) + O(\varepsilon^2). \tag{B4}$$

The remaining trigonometric terms vanish by averaging over the fast oscillations, and one obtains

$$I_1(t) \approx -T^2 b_y'(t), \tag{B5}$$

$$I_2(t) \approx -Tb_x(t). \tag{B6}$$

APPENDIX C: COMBINED AVERAGES WITHIN THE CORRSIN APPROXIMATION

For the evaluation of the Lagrangian velocity correlation we perform the following calculation which is based on the cumulant expansion with Gaussian statistics:

$$\begin{aligned} &\left\langle a(t_1)a(t_2)\exp\left(-ik\int_{t_2}^{t_1} a(t')dt'\right)\right\rangle \\ &= \frac{1}{k^2}\frac{\partial^2}{\partial t_1\,\partial t_2}\left\langle \exp\left(-ik\int_{t_2}^{t_1} a(\tau)d\tau\right)\right\rangle \\ &\approx \frac{1}{k^2}\frac{\partial^2}{\partial t_1\,\partial t_2}\exp\left(-\frac{1}{2}k^2\int_{t_2}^{t_1}\int_{t_2}^{t_1}\langle a(\tau_1)a(\tau_2)\rangle d\tau_1 d\tau_2\right) \\ &= (\langle a(t_1)a(t_2)\rangle - k^2\varphi_a^2)\exp\left(-\frac{1}{2}k^2\psi_a\right). \end{aligned} \tag{C1}$$

The functions φ_a and ψ_a are defined similarly to Eqs. (32) and (33). The procedure (C1) allows us to evaluate the combined average which appears within the calculation of the Lagrangian velocity correlators. An equivalent average involving a series expansion of the exponential function was used in Ref. [15].

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